

# Digital Control (Session 1)

## \* Review (Control Engineering)

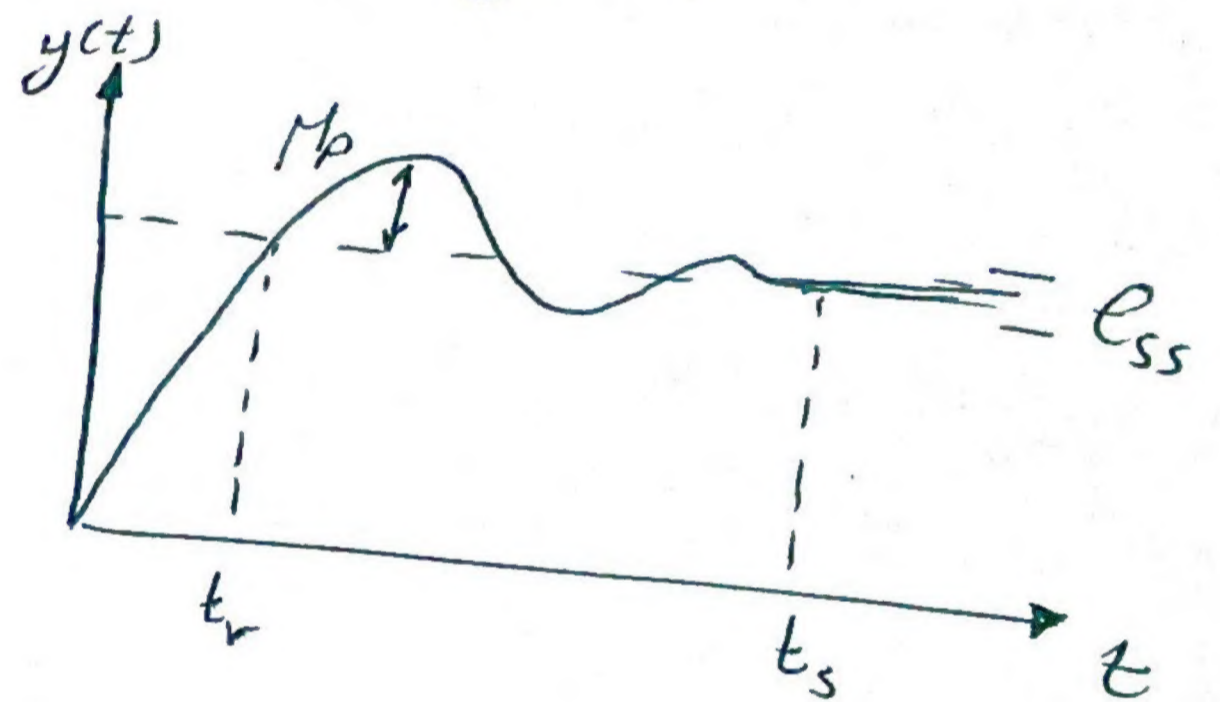
- Control theory is the general framework for studying dynamical systems
- Control Systems Objective is to improve the behaviour of the system to meet design specs (improve dynamics - reduce error)
- Any Control System Response Can be characterized by 4-parameters:

$t_s$ : settling time

$t_r$ : rise time

$M_p$ : Maximum Overshoot

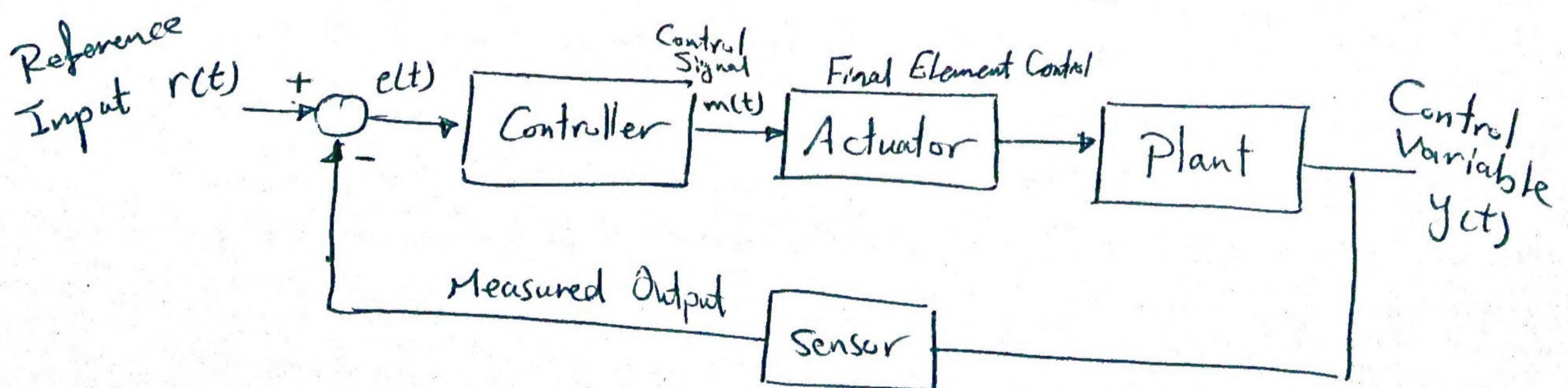
$e_{ss}$ : Steady State Error



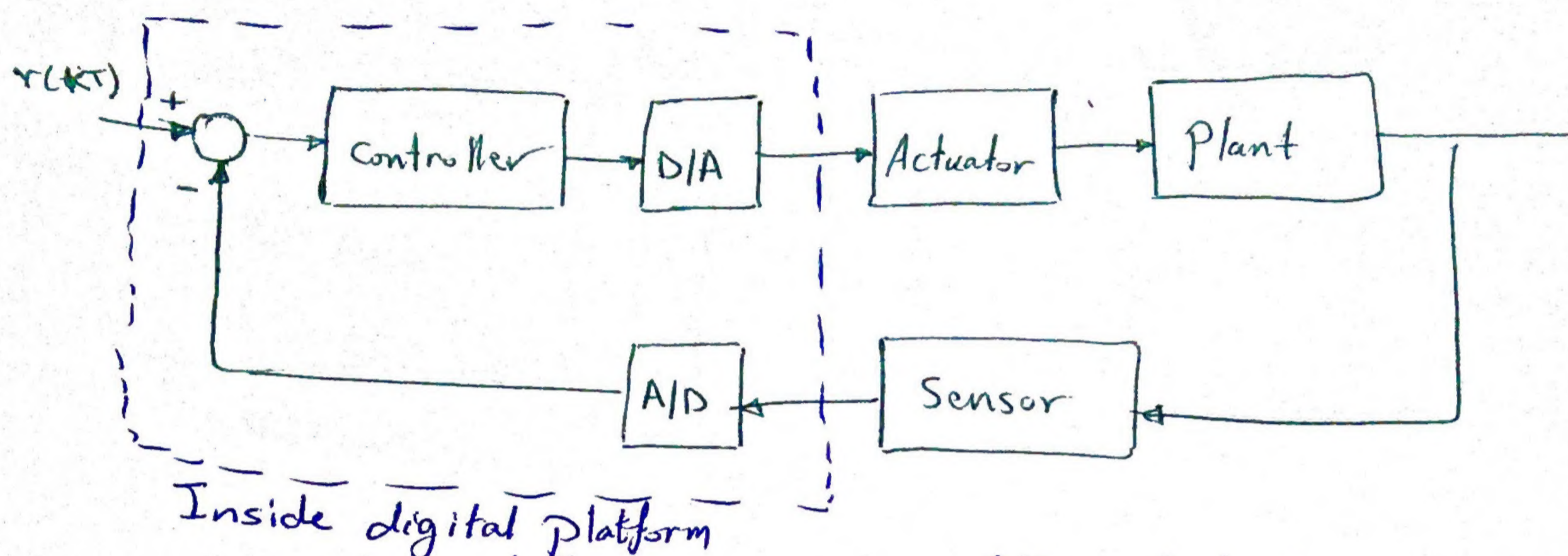
## - The Phases of Control System design

- Plant Modeling (differential Equations - Transfer function)
- System Analysis (Root Locus - Frequency Response)
- Controller design (PID - Lead/Lag Compensator - State feedback)
- Implementation (Analog Components (op-amps + RLC circuits))

## - The Block diagram of a Feedback Control System



## \* Digital Control System Structure



Inside digital platform

- Microcontroller
- DSP
- Computer
- PLC

A/D: Analog to Digital Converter

D/A: Digital to Analog Converter

\* Digital platforms are the most powerful computing platforms specially with today's high speed of processors (much more than speed of systems to be controlled).

## \* Digital Control Benefits:

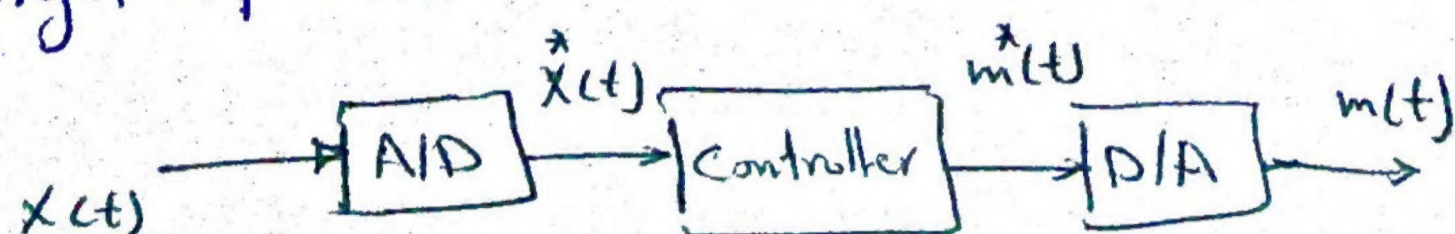
- Control Algorithm is converted to a code on a computer system
- More powerful to implement complex control algorithms (there are some control algorithms that can be implemented only using digital control techniques)
- Reduce the need of analog components (affected by noise)
- More efficient to be modified and scaled
- More robust to environment disturbance

## \* Digital Control challenges:

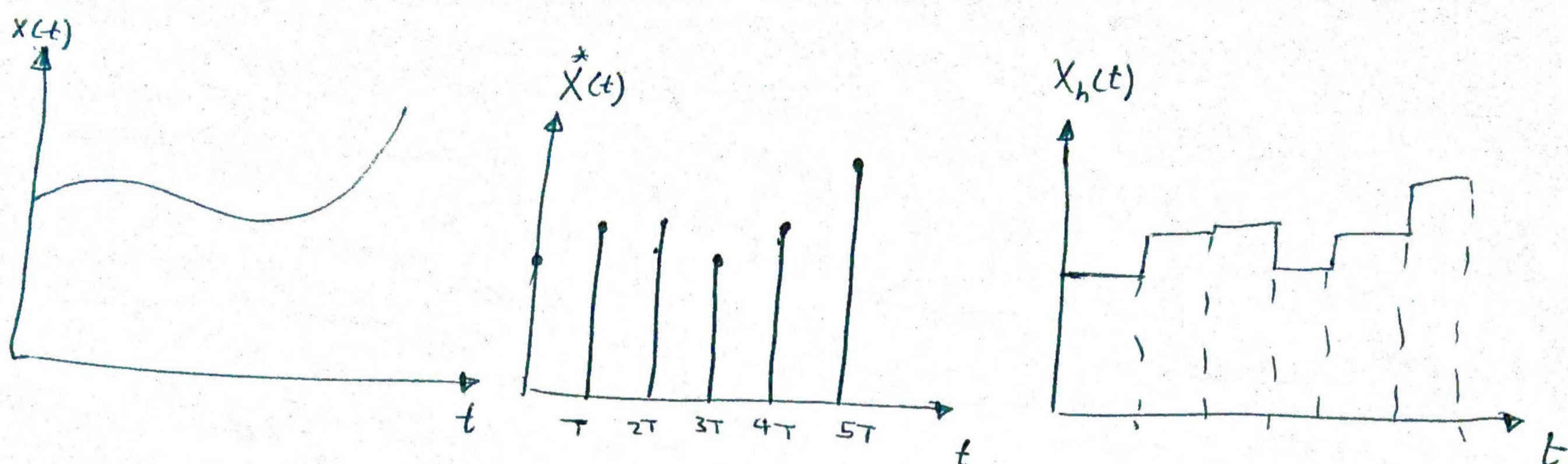
- Sampling Rate
- Reconstruction of original signals

As physical systems are continuous by nature we approximate them by discretization

## \* Digital plant:



## \* Sampling and Reconstruction



$x(t)$  : Original Analog Signal

$x^*(t)$  : Sampled Signal

$x_h(t)$  : Reconstructed Signal

$x_h(t)$  : the reconstructed signal using  
Holder circuit is a ladder signal  
It can be smoothed using  
Low Pass Filter

We note that

$$\begin{aligned} x^*(t) &= \sum_{k=0}^{\infty} \delta(t - kT) x(t) \quad , T: \text{Sampling Period} \\ &= \delta(t) x(0) + \delta(t - T) x(T) + \dots \\ &= x(0) + x(T) + x(2T) + \dots \end{aligned}$$

Remember Transfer Functions describing Systems using Laplace Transform

$$\mathcal{L}[x^*(t)] = \sum_{k=0}^{\infty} x(kT) e^{-kTS} \quad (\text{Integration becomes Summation})$$

Note : This is the definition of Z-transform

$$Z = e^{kTS} \quad (Z: \text{advance operator} \\ Z^{-1}: \text{delay operator})$$

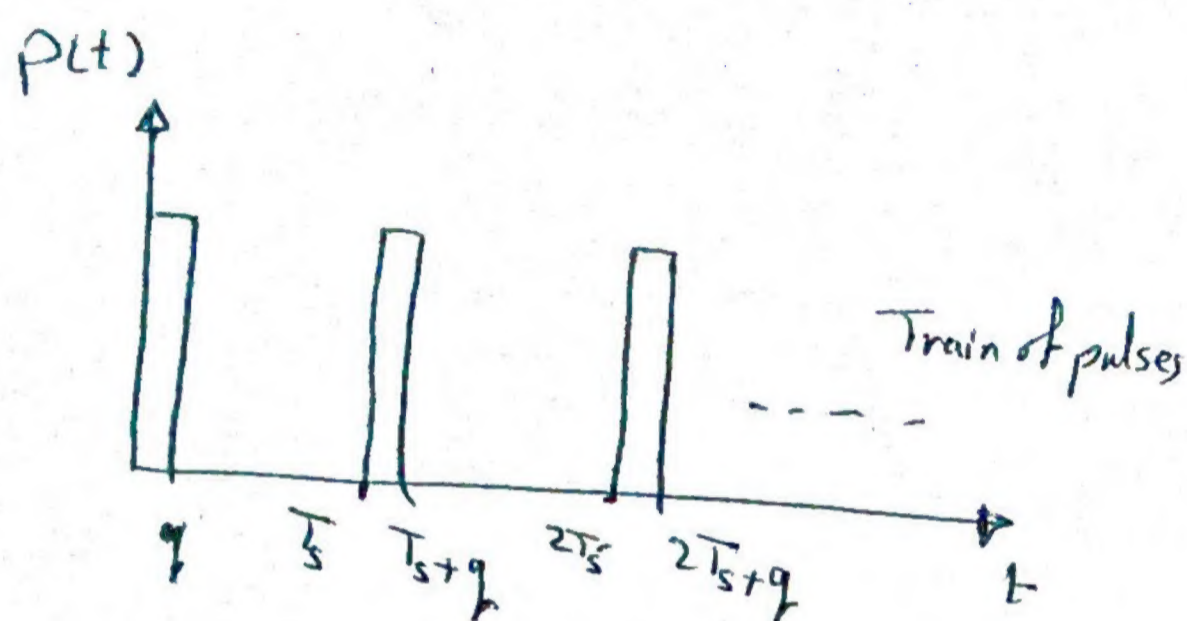
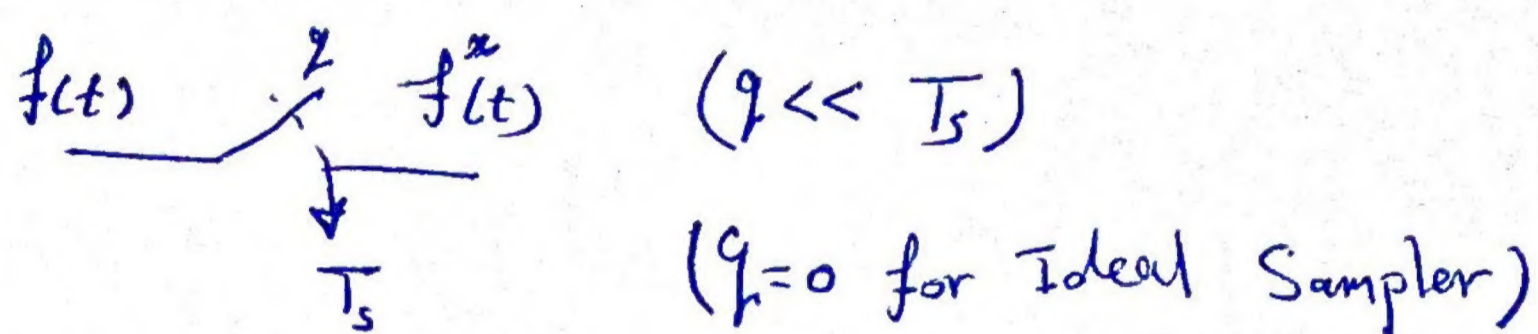
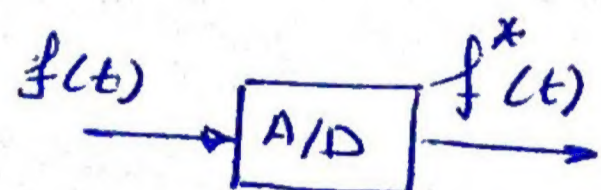
$\therefore$  We need to make revision on Z-transform

- \* Z-transform represents discrete Systems Transfer function
- \* Difference equations are solved using Z-transform

## \* Sampling Process :-

→ Is there a relation between Sampling frequency and Signal frequency?

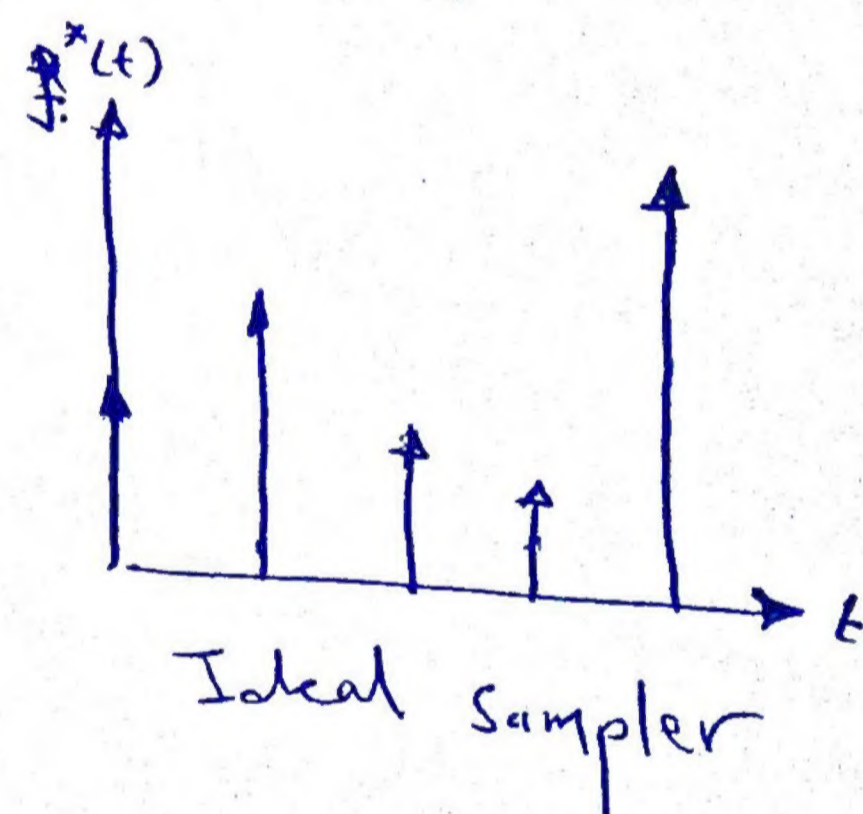
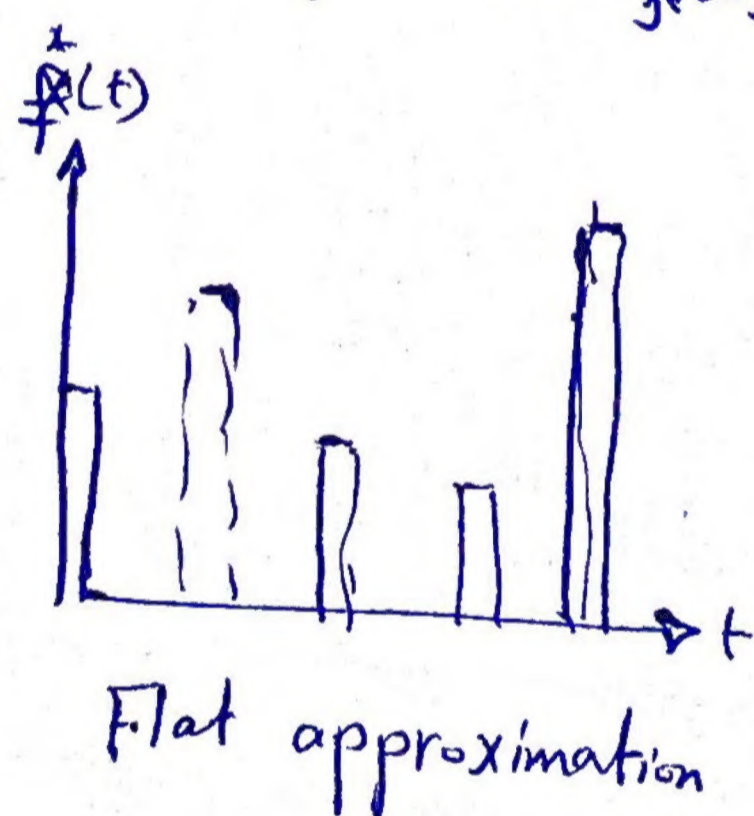
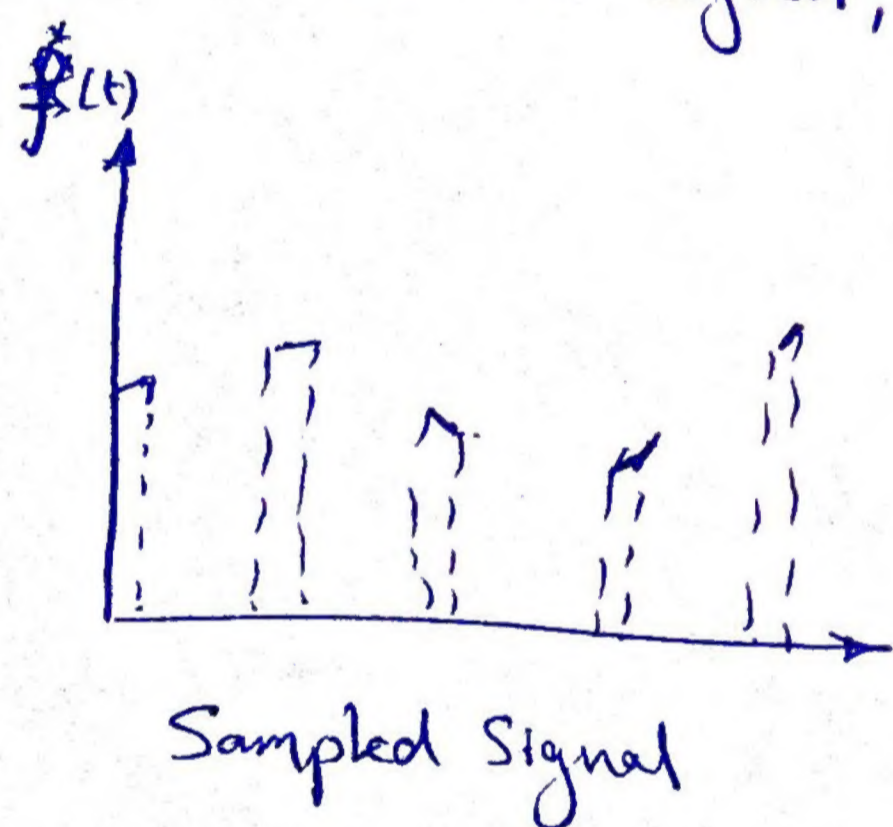
For A/D Converter:



$p(t)$  : models the opening / closing of Sampler Switch

then,  $x^*(t)$  is the result of multiplying  $x(t)$  by  $p(t)$

\* The Sampler is considered as an amplitude Modulation device with  $p(t)$  as the Carrier Signal,  $f(t)$  the input and  $f^*(t)$  is the output signal.



$$f^*(t) = p(t) f(t)$$

$$(q \ll T_s)$$

where

$p(t)$  : periodic function

$$p(t) = \sum_{k=-\infty}^{\infty} [U(t - kT) - U(t - (kT + q))]$$

$q$  :  $t_{on}$

$T_s$  : Sampling Period

\* The relation between  $f_s$  and  $f_o$

$f_s$ : Sampling frequency

$f_o$ : Signal frequency

We use Fourier Analysis to derive the relation between  $f_s$  and  $f_o$ .

① Express  $P(t)$  using Fourier Series ( $P(t)$  is periodic)

$$P(t) = \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)$$

$$C_n = \frac{1}{T} \int_0^T P(t) \exp(-jn\omega_s t) dt$$

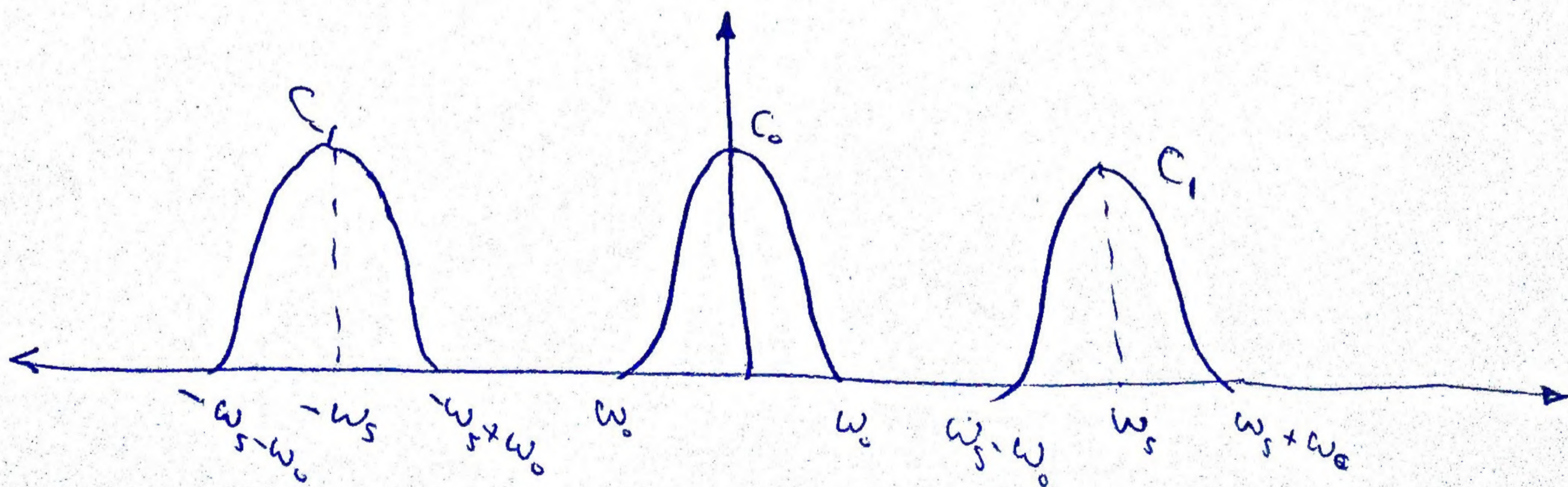
$$\therefore f^*(t) = f(t) \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)$$

② Taking Fourier transform of  $f^*(t)$  (Sampled Signal)

$$\begin{aligned} F(f^*(t)) &= F^*(\omega) = F\left[f(t) \sum_{n=-\infty}^{\infty} C_n \exp(jn\omega_s t)\right] \\ &= \sum_{n=-\infty}^{\infty} C_n F(\exp(jn\omega_s t) f(t)) \end{aligned}$$

but  $\exp(jn\omega_s t) f(t) \xrightarrow{F} F(j\omega - jn\omega_s)$

$$\therefore F^*(\omega) = \sum_{n=-\infty}^{\infty} C_n F(j\omega - jn\omega_s)$$



From Spectrum of discrete (Sampled Signal):

⑥

① if  $\omega_s > 2\omega_0$

we LPF  $\rightarrow$  Restore  $f(t)$  ✓

② if  $\omega_s = 2\omega_0$

Critical case for restoring  $f(t)$

③ if  $\omega_s < 2\omega_0$

we can't Reconstruct the Signal

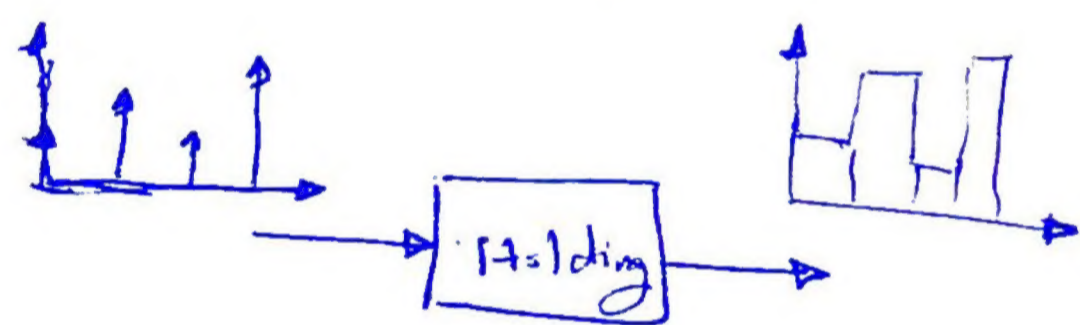
Shannon's Theory 2

For right sampling Process  $\omega_s$  (sampling frequency) must be equal or larger than twice  $\omega_0$

$$\omega_s \geq 2\omega_0$$

1) In Practical  $\omega_s$  is chosen to be  $(5-10) \cdot \omega_0$

\* Holding Process



$$f_k(t) = f(kT) + f'(kT)(t-kT) + f''(kT) \frac{(t-kT)^2}{2!} + \dots$$

Taylor Expansion

where

-  $f_k(t)$ : The function expression in between  $(kT)$  and  $(kT+T)$

-  $f(kT) = f(t) |_{t=kT}$ ,  $f'(kT) = \frac{d}{dt} f(t) |_{t=kT}$

$\rightarrow$  If we choose only first term  $\rightarrow f_k(t) = f(kT)$ : Zero Order Hold



$$\therefore G_{ZOH}(s) = \frac{O/P(s)}{I/P(s)} = \frac{\frac{1}{s} - e^{-Ts} \cdot \frac{1}{s}}{1} = \boxed{\frac{1 - e^{-Ts}}{s}}$$